Regularization of the Burnett Equations via Relaxation

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The classical Chapman–Enskog expansions for the pressure deviator **P** and heat flux **q** provide a natural bridge between the kinetic description of gas dynamics as given by the Boltzmann equation and continuum mechanics as given by the balance laws of mass, momentum, energy supplemented by the expansions for **P** and **q**. Truncation of these expansions beyond first (Navier–Stokes) order yields instability of the rest state and is inconsistent with thermodynamics. In this paper we propose a visco-elastic relaxation approximation that eliminates the instability paradox. This system is weakly parabolic, has a linearly hyperbolic convection part, and is endowed with a generalized entropy inequality. It agrees with the solution of the Boltzmann equation up to the Burnett order via the Chapman– Enskog expansion.

0. INTRODUCTION

The classical Chapman–Enskog procedure for the Boltzmann equation[5] is a well known tool for bridging the gap between kinetic theory as described by the Boltzmann equation for the evolution of a monatomic gas and

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continuum mechanics. The Chapman–Enskog expansion is a formal power series ordered by the viscosity μ which is itself proportional to the non-dimensional Knudsen number, i.e.,

$$\mathbf{T} = -p\mathbf{I} - \mathbf{P}$$
$$p = R\rho\theta$$
$$\mathbf{P} = \mu\mathbf{\Pi}^{(1)} + \mu^2\mathbf{\Pi}^{(2)} + \mu^3\mathbf{\Pi}^{(3)} + \cdots$$
$$\mathbf{q} = \mu\mathbf{\Xi}^{(1)} + \mu^2\mathbf{\Xi}^{(2)} + \mu^3\mathbf{\Xi}^{(3)} + \cdots \tag{0.1}$$

The coefficients $\mathbf{\Pi}^{(j)}$, $\mathbf{\Xi}^{(j)}$, $j = 1, 2, \ldots$ are obtained from the Boltzmann equation and have been determined up to $j = 2$ (Burnett order)[8, 23] and in one space dimension up to $j = 3$ (super-Burnett order).[10] (We remind the readers that all physical quantities in this paper and their mathematical definitions are given in the Nomenclature.)

In practice however the Chapman–Enskog expansion as a tool for solving the Boltzmann equation has had limited practical value. Truncation at first order yields the Navier–Stokes equations which as μ ceases to be small becomes a poor approximation to solutions of the Boltzmann equation.[15, 19] Truncation at order μ^2 yields the Burnett equations which possesses the unphysical property of yielding linearly unstable rest states.[1, 3, 16–18] Simply by expanding to the higher order will not remove this instability.[20] In addition, the Chapman–Enskog expansion destroys the material frame indifference at the Burnett order.[2]

Despite the linear instability of the Burnett equations, numerical solutions on augmented Burnett equations[1, 9, 23] suggest that they provide more accurate solutions in the shock layer than those of the Navier–Stokes equations when compared with the direct simulation Monte-Carlo method of the Boltzmann equation. In ref. 1, 9, 23 the augmented Burnett equations were obtained either by removing the unstable term from or by adding linearly stabilizing terms of the super Burnett order to the stress and heat flux. Unfortunately the augmented Burnett equations possess two drawbacks: (i) Numerically they require resolution of the super-Burnett stabilizing terms which practically means numerical resolution of derivates up to fourth order. This is rather a cumbersome approach in several space dimensions; (ii) the augmented Burnett equations have not been shown to have a globally defined "entropy" possessing the usual property of satisfying an "entropy" inequality.

In this paper we propose a visco-elastic regularization that (i) requires at most resolution of second derivatives in spatial variables; (ii) possesses a

globally defined "entropy" like function; (iii) still has the property that our system when expanded via the Chapman–Enskog expansion still matches the classical Chapman-Enskog expansion for the Boltzmann equation to the Burnett order. Specifically we relax the pressure deviator and heat flux by rate equations to obtain a system of local equations that can recover the Burnett equations via the Chapman–Enskog expansion with a correction at the super-Burnett order. By doing this, we obtain a system of thirteen local equations that is linearly stable. This system is weakly parabolic with a linearly hyperbolic convection part. Moreover, it is endowed with a generalized entropy inequality. The nonlinear entropy inequality guarantees the irreversibility of the relaxation process. The localness of this system is attractive for a robust numerical approximation to the gas dynamics valid to the Burnett order.

Relaxation as a stabilizing mechanism is well known in materials of rate type where the stress and heat flux satisfy separate evolution equations. (see for example ref. 13). In fact it is precisely asymptotic expansions such as the Chapman–Enskog expansion that formally elicits this feature.[6] In recent years, relaxation approximations have been used as an effective tool to design numerical methods—known as the relaxation schemes. In ref. 12 a generic way to relax a general system of hyperbolic conservation laws was introduced by Jin and Xin, which induced a class of relaxation schemes free of Riemann solver and local characteristic decomposition for inviscid gas dynamics. A physically natural pressure relaxation method was developed by Coquel and Perthame for an inviscid general gas.[7]

Our method differs from the classical approaches of Grad,[11] Levermore[14, 15] and the extended thermodynamics.[19] For example, as noted above, our thirteen equations asymptotically match the Chapman–Enskog expansion to the Burnett order, while in comparison the extended Grad moment closures need twenty-six moments to fulfill the same task.[22]

With regard to the issue of material frame indifference we point out that the lost material frame indifference in the Chapman–Enskog expansion cannot be recovered via relaxation, nor can it be by, for example, Grad's theory. However, we do not view this as a serious defect in our theory, since our goal is to develop a mathematical algorithm to approximate the Boltzmann equation.

Many questions remain to be answered along this line of research. For example, both analytical (in 1-D) and numerical study of the structure of the shock profile, and its comparison with the Navier–Stokes profile and that of the Direct Simulation Monte-Carlo solution, are necessary to justify the value of this work. Another subtle issue is the boundary condition for the relaxation system. These questions are currently under investigation by the authors.

The paper is divided into five sections after this Introduction. Section 1 reviews the balance laws of mass, momentum, energy, and the entropy production—the Clausius–Duhem inequality for continuum fluid dynamics. Section 2 presents the Chapman–Enskog expansion for pressure deviator **P** and heat flux **q** up to $n = 2$, the Burnett order. In Section 3 we introduce and study the relaxation approximation. In Section 4 we prove a generalized entropy inequality for the relaxation system, while Section 5 is contributed to the study of hyperbolicity of the linearized relaxation system.

1. THE FIELD EQUATIONS OF BALANCE

The field equations of balance for continuum fluid dynamics in the absence of heat sources are as follows:

$$\dot{\rho} + \rho \operatorname{div} \mathbf{u} = 0, \qquad \text{(mass conservation)} \tag{1.1}$$

$$\rho \dot{\mathbf{u}} + \operatorname{grad} p + \operatorname{div} \mathbf{P} = \rho \mathbf{b}, \qquad \text{(linear momentum conservation)} \tag{1.2}$$

$$\mathbf{P} = \mathbf{P}^T, \qquad \text{(rotational momentum conservation)} \tag{1.3}$$

$$\rho \dot{\mathrm{e}} + p \operatorname{div} \mathbf{u} + \mathbf{P} \cdot \mathbf{S} + \operatorname{div} \mathbf{q} = 0, \qquad \text{(energy conservation)} \tag{1.4}$$

where

$$\mathrm{e} = \psi - \theta \frac{\partial \psi}{\partial \theta}, \qquad \eta = -\frac{\partial \psi}{\partial \theta}, \qquad p = \rho^2 \frac{\partial \psi}{\partial \rho} \tag{1.5}$$

Differentiation of the expression for the Helmholtz free energy $\psi = \varepsilon - \theta \eta$ yields

$$\rho \theta \dot{\eta} = \rho \dot{\mathrm{e}} - \rho \dot{\rho} \frac{\partial \psi}{\partial \rho} \tag{1.6}$$

which when combined with (1.1), (1.4), (1.6) yields the entropy production equation

$$\rho \theta \dot{\eta} = -\mathbf{P} \cdot \mathbf{S} - \operatorname{div} \mathbf{q} \tag{1.7}$$

Division by θ yields the total entropy product rate of a fluid occupying domain $\mathscr{B} \subset \mathbb{R}^3$:

$$\frac{d}{dt} \int_{\mathscr{B}} \rho \eta \, dV = -\int_{\mathscr{B}} \frac{\mathbf{P} \cdot \mathbf{S}}{\theta} + \frac{\mathbf{q} \cdot \operatorname{grad} \theta}{\theta^2} \, dV - \int_{\partial \mathscr{B}} \frac{\mathbf{q} \cdot \mathbf{n}}{\theta} \, dA \tag{1.8}$$

The Clausius–Duhem inequality is a common albeit not universally accepted form of the second law of thermodynamics. It asserts

$$\frac{d}{dt} \int_{\mathscr{B}} \rho\eta \, dV + \int_{\partial\mathscr{B}} \frac{\mathbf{q} \cdot \mathbf{n}}{\theta} \, dA \geqslant 0 \tag{1.9}$$

which in turn from (1.8) requires $\mathbf{P}$, $\mathbf{q}$ to satisfy

$$\int_{\mathscr{B}} \frac{\mathbf{P} \cdot \mathbf{S}}{\theta} + \frac{\mathbf{q} \cdot \operatorname{grad} \theta}{\theta^2} \, dV \leqslant 0 \tag{1.10}$$

for all fluid domains $\mathscr{B}$. However the classical Clausius–Duhem inequality is inconsistent with $\mathbf{P}$, $\mathbf{q}$ delivered by the Chapman–Enskog expansion beyond Navier–Stokes order.

2. THE CHAPMAN–ENSKOG EXPANSION

The Chapman–Enskog expansion for a monatomic gas of spherical molecules yields the constitutive relations

$$
\begin{aligned}
&\mathrm{e} = \tfrac{3}{2} R\theta, \qquad p = R\rho\theta, \qquad \mu = \mu(\theta) \\
&\psi = R\theta \log \rho - \tfrac{3}{2} R\theta \log + \tfrac{3}{2} R\theta - a\theta + b \\
&\eta = -R \log \rho + \tfrac{3}{2} R \log \theta + a
\end{aligned} \tag{2.1}
$$

where a, b are constants of integration.

In addition the expansion provides representations for the pressure deviator tensor $\mathbf{P}$ and heat flux vector $\mathbf{q}$ in terms of a series which may be ordered via powers of the viscosity μ in terms of the total number of space plus time derivatives. Following the notation of Ferziger and Kaper[8] we record

$$\mathbf{P} = \mu\mathbf{P}^{(1)} + \mu^2\mathbf{P}^{(2)} + \cdots \tag{2.2}$$

$$\mathbf{q} = \mu\mathbf{q}^{(1)} + \mu^2\mathbf{q}^{(2)} + \cdots \tag{2.3}$$

where the expressions for $\mathbf{P}^{(1)}$, $\mathbf{P}^{(2)}$, $\mathbf{q}^{(1)}$, $\mathbf{q}^{(2)}$ are as follows:

$$\mathbf{P}^{(1)} = -2\mathbf{S} \tag{2.4}$$

$$\mathbf{q}^{(1)} = -\tfrac{3}{2} MR \operatorname{grad} \theta \tag{2.5}$$

$$\mathbf{P}^{(2)} = \omega_1 \frac{1}{p} (\text{div } \mathbf{u}) \, \mathbf{S}$$

$$+ \omega_2 \frac{1}{p} \left\{ \dot{\mathbf{S}} - \mathbf{LS} - \mathbf{SL}^{\mathsf{T}} + \frac{2}{3} tr(\mathbf{SL}^{\mathsf{T}}) \, \mathbf{I} \right\}$$

$$+ \omega_3 \frac{1}{\rho\theta} \left\{ \text{grad}^2 \, \theta - \frac{1}{3} \Delta\theta\mathbf{I} \right\}$$

$$+ \omega_4 \frac{1}{\rho p \theta} \left\{ \frac{1}{2} \text{grad } p \otimes \text{grad } \theta \right.$$

$$+ \frac{1}{2} \text{grad } \theta \otimes \text{grad } p - \frac{1}{3} \text{grad } p \cdot \text{grad } \theta\mathbf{I} \right\}$$

$$+ \omega_5 \frac{1}{\rho\theta^2} \left\{ \text{grad } \theta \otimes \text{grad } \theta - \frac{1}{3} |\text{grad } \theta|^2 \, \mathbf{I} \right\}$$

$$+ \omega_6 \frac{1}{p} \left\{ \mathbf{S}^2 - \frac{1}{3} tr(\mathbf{S}^2) \, \mathbf{I} \right\} \qquad (2.6)$$

$$\mathbf{q}^{(2)} = \theta_1 \frac{1}{\rho\theta} (\text{div } \mathbf{u}) \, \text{grad } \theta$$

$$+ \theta_2 \frac{1}{\rho\theta} ((\text{grad } \theta)^{\cdot} - \mathbf{L}^T \, \text{grad } \theta)$$

$$+ \theta_3 \frac{1}{p\rho} (\mathbf{S} \, \text{grad } p)$$

$$+ \theta_4 \frac{1}{\rho} \text{div } \mathbf{S} + \theta_5 \frac{1}{\rho\theta} \mathbf{S} \, \text{grad } \theta \qquad (2.7)$$

One drawback of the Chapman–Enskog expansion is that, if truncated at the Burnett or higher order, it destroys the property of material frame indifference. In particular, in (2.6) and (2.7), the ω^2 term in $\mathbf{P}^{(2)}$ and the θ_2 term in $\mathbf{q}^{(2)}$ are both material frame different. It cannot be recovered by replacing the material derivative with the space derivative using the Euler or Navier–Stokes equations.[2]

The coefficients $\omega_1, ..., \omega_6, \theta_1, ..., \theta_5$ are functions of θ and are not independent. For a gas of spherical molecules the following universal relations have been derived by Truesdell and Muncaster[23] generalizing more specialized relations:

$$\omega_3 = \theta_4$$

$$\theta_1 = \frac{2}{3}\left(\frac{7}{2} - \frac{\mu'(\theta)}{\mu(\theta)}\theta\right)\theta_2 - \frac{1}{3}\theta\frac{\partial\theta_2}{\partial\theta}$$

$$\omega_1 = \frac{2}{3}\left(\frac{7}{2} - \frac{\mu'(\theta)}{\mu(\theta)}\theta\right)\omega_2 - \frac{1}{3}\theta\frac{\partial\omega_2}{\partial\theta} \qquad (2.8)$$

Furthermore for gases of ideal spheres in which the collisions are purely elastic or satisfy an inverse kth-power attraction between molecules, the coefficients $\omega_1, \omega_2, ..., \theta_5$ are independent of θ. In addition the relations

$$\frac{\theta_1}{\theta_2} = \frac{\omega_1}{\omega_2} = \begin{cases} \dfrac{2}{3}\left(\dfrac{3k-5}{k-1}\right) & \text{for inverse } k\text{th power molecules} \\ 2 & \text{for ideal spheres} \end{cases} \qquad (2.9)$$

hold.

Exact determination of $\omega_1, \omega_2, ..., \theta_5$ has only been accomplished for a gas of Maxwellian ($k=5$) molecules. For the more general case only approximations to $\omega_1, \omega_2, ..., \theta_5$ have been obtained. The classical approximation result (say as found in Ferziger and Kaper [8, p. 149]) is

$$\omega_2 \simeq 2, \qquad \omega_3 \simeq 3, \qquad \omega_4 \simeq 0, \qquad \omega_5 \simeq \frac{\mu'(\theta)\,\theta\omega_3}{\mu(\theta)}, \qquad \omega_6 \simeq 8$$

$$\theta_2 \simeq \frac{45}{8}, \qquad \theta_3 \simeq -3, \qquad \theta_4 \simeq 3, \qquad \theta_5 \simeq 3\left(\frac{35}{4} + \frac{\theta}{\mu}\mu'(\theta)\right)$$

$$M \simeq \frac{5}{2} \qquad (2.10)$$

For Maxwell molecules the relations 2.10 are exact: $\frac{\theta_1}{\theta_2} = \frac{\omega_1}{\omega_2} = \frac{5}{3}$, $\frac{\theta\mu'(\theta)}{\mu(\theta)} = 1$, and μ is linear in θ.

In this paper we shall assume that in addition to (2.8) the following relations hold

$$\theta_3 + \omega_3 + \omega_4 = 0$$

$$\omega_5 = \frac{\mu'(\theta)\,\theta}{\mu(\theta)}\omega_3$$

$$\theta_5 = \bar{\theta}_5 + \frac{\mu'(\theta)\,\theta}{\mu(\theta)}\omega_3$$

$$\omega_3 > 0, \qquad \theta_2 > 0$$
$$\bar{\theta}_5 > 0, \qquad \bar{\theta}_5 \qquad \text{a constant} \tag{2.11}$$

Notice the assumption (2.11) holds for the approximation (2.10) but does not assume the molecules are ideal spheres or satisfy an inverse kth power attraction law. Of course (2.8), (2.11) are satisfied by Maxwell molecules. However we reiterate the fact that relation (2.8) and the first equation in (2.11) are universal for all spherical molecules.[25]

3. A RELAXATION APPROXIMATION

Since it is the material derivative terms on the right hand side of (2.6) and (2.7) that introduce the linear instability,[21] we seek a relaxation approximation that regularizes $\mathbf{P}$ and $\mathbf{q}$. Specifically, we write

$$\dot{\rho} + \rho \operatorname{div} \mathbf{u} = 0 \tag{3.1}$$

$$\rho \dot{\mathbf{u}} + \operatorname{grad} p + \operatorname{div} \mathbf{P} = \rho \mathbf{b} \tag{3.2}$$

$$\mathbf{P} = \mathbf{P}^T \tag{3.3}$$

$$\dot{\mathbf{P}} - \mathbf{L} \mathbf{P} - \mathbf{P} \mathbf{L}^T + \frac{2}{3} \operatorname{tr}(\mathbf{P} \mathbf{L}^T) \, \mathbf{I} = -\frac{2p}{\omega_2 \mu} (\mathbf{P} - \mathbf{P}^{eq}) \tag{3.4}$$

$$\rho \, \dot{e} + p \operatorname{div} \mathbf{u} + \mathbf{P} \cdot \mathbf{S} + \operatorname{div} \mathbf{q} = 0 \tag{3.5}$$

$$\dot{\mathbf{q}} - \mathbf{L}^T \mathbf{q} = -\frac{3Mp}{2\theta_2 \mu} (\mathbf{q} - \mathbf{q}^{eq}) \tag{3.6}$$

where

$$\mathbf{P}^{eq} = -2\mu \mathbf{S} + \mathbf{P}_2 + \mathbf{P}_3 \tag{3.7}$$

$$\mathbf{P}_2 = -\mu \frac{\omega_1}{2p} (\operatorname{div} \mathbf{u}) \, \mathbf{P} + \frac{\omega_2 \mu'(\theta) \, \dot{\theta}}{2p} \, \mathbf{P}$$

$$+ \mu^2 \frac{\omega_3}{\rho \theta} \left\{ -\operatorname{grad} \left(\frac{\mathbf{q}}{\frac{3}{2} \mu M R} \right) + \frac{1}{3} \operatorname{div} \left(\frac{\mathbf{q}}{\frac{3}{2} \mu M R} \right) \mathbf{I} \right\}$$

$$+ \mu \frac{\omega_4}{\rho p \rho} \left\{ -\frac{1}{2} \operatorname{grad} p \otimes \left(\frac{\mathbf{q}}{\frac{3}{2} MR} \right) - \frac{1}{2} \left(\frac{\mathbf{q}}{\frac{3}{2} MR} \right) \times \operatorname{grad} p \right.$$

$$\left. + \frac{1}{3} \operatorname{grad} p \cdot \left(\frac{\mathbf{q}}{\frac{3}{2} MR} \right) \mathbf{I} \right\}$$

$$- \mu \frac{\omega_5}{\rho \theta^2} \left\{ \frac{1}{2} \operatorname{grad} \theta \otimes \left(\frac{\mathbf{q}}{\frac{3}{2} MR} \right) + \frac{1}{2} \left(\frac{\mathbf{q}}{\frac{3}{2} MR} \right) \otimes \operatorname{grad} \theta \right.$$

$$\left. - \frac{1}{3} \operatorname{grad} \theta \cdot \left(\frac{\mathbf{q}}{\frac{3}{2} MR} \right) \mathbf{I} \right\} - \mu \frac{\omega_6}{2p} \left\{ \frac{1}{2} (\mathbf{SP} + \mathbf{PS}) - \frac{1}{3} \operatorname{tr}(\mathbf{PS}) \mathbf{I} \right\} \tag{3.8}$$

$$\mathbf{P}_3 = \mu^2 \left[\frac{\hat{\omega}_2}{p^2} \operatorname{tr} \mathbf{S}^2 + \hat{\omega}_3 \frac{|\operatorname{grad} \theta|^2}{R \rho^2 \theta^3} \right] \mathbf{P} + \mu \frac{\hat{\gamma}_1}{p \theta} \left(\dot{\theta} + \frac{2}{3} \theta \operatorname{div} \mathbf{u} \right) \mathbf{P}$$

$$+ \hat{\omega}_4 \left[\frac{\mu^3}{MR\rho^2} \left(\frac{1}{2\mu\theta} P^{ij} \right)_{,k} \right]_{,k} \tag{3.9}$$

$$\mathbf{q}^{eq} = -\frac{3}{2} \mu MR \operatorname{grad} \theta + \mathbf{q}_2 + \mathbf{q}_3 \tag{3.10}$$

$$\mathbf{q}_2 = -2\mu \frac{\theta_1}{3MR\rho\theta} (\operatorname{div} \mathbf{u}) \mathbf{q} + \frac{2\theta_2 \dot{\theta} \mu'(\theta)}{3MR\rho\theta} \mathbf{q}$$

$$- \mu \frac{\theta_3}{2p\rho} \mathbf{P} \operatorname{grad} p - \mu^2 \frac{\theta_4}{2\rho} \operatorname{div} \left(\frac{\mathbf{P}}{\mu} \right) - \mu \frac{\theta_5}{2\rho\theta} \mathbf{P} \operatorname{grad} \theta \tag{3.11}$$

$$\mathbf{q}_3 = \mu^2 \left[\frac{\hat{\theta}_2}{p^2} \operatorname{tr} \mathbf{S}^2 + \hat{\theta}_3 \frac{|\operatorname{grad} \theta|^2}{R\rho^2\theta^3} \right] \mathbf{q} + \mu \frac{\hat{\lambda}_1}{\rho\theta^2} \left(\dot{\theta} + \frac{2}{3} \theta \operatorname{div} \mathbf{u} \right) \left(\frac{\mathbf{q}}{\frac{3}{2} MR} \right)$$

$$+ \hat{\theta}_4 \left[\frac{\mu^3 \theta}{\rho^2} \left(\frac{2}{3MR\mu\theta^2} q_i \right)_{,k} \right]_{,k} \tag{3.12}$$

In (3.10) and (3.12) conventional summation notation is used. Since the energy equation (3.5) implies that

$$\dot{\theta} + \frac{2}{3} \theta \operatorname{div} \mathbf{u} = \frac{2}{3\rho R} (-\mathbf{P} \cdot \mathbf{S} - \operatorname{div} \mathbf{q}) \tag{3.13}$$

system (3.1)–(3.6) is weakly parabolic and local (does not contain $\dot{\theta}$ on the right hand side) after using (3.13). Moreover, (3.13) suggests that

$\dot{\theta} + \frac{2}{3}\theta$ div $\mathbf{u} = O(\mu)$, and $\mathbf{P}_3$ and $\mathbf{q}_3$ are $O(\mu^3)$, thus belong to the super Burnett order. It is a trivial observation that (3.1)–(3.6) yield a representation of $\mathbf{P}, \mathbf{q}$ in powers of μ, which agrees with the classical Chapman–Enskog expansion (2.2)–(2.7) to Burnett order, i.e., terms of order μ^2. Yet unlike the augmented Burnett systems of refs. 1, 9, 23 the system possesses spatial derivatives only up to second order.

4. A GENERALIZED ENTROPY INEQUALITY

We shall prove a generalized entropy inequality (Theorem 4.2) for the relaxation systems (3.1)–(3.7). This inequality guarantees the irreversibility of the relaxation process. In addition to the classical entropy for the Navier–Stokes equations, the generalized entropy also depends on the nonequilibrium variables $\mathbf{P}$ and $\mathbf{q}$.

Lemma 4.1. Let $\mathbf{P}, \mathbf{q}$ be given by (3.1)–(3.12) with

$$
\begin{aligned}
\hat{\lambda}_1 &= -\frac{1}{2}\theta\frac{\partial\theta_2}{\partial\theta} - \theta_2\theta\frac{\mu'(\theta)}{\mu(\theta)} + \frac{3}{2}\theta_2 \\
\hat{\gamma}_1 &= -\frac{1}{2}\theta\frac{\partial\omega_2}{\partial\theta} - \omega_2\theta\frac{\mu'(\theta)}{\mu(\theta)} + \omega_2
\end{aligned}
\tag{4.1}
$$

in (3.9), (3.12) respectively. Then the following equality holds:

$$
\begin{aligned}
&\rho\left\{-\eta + \frac{1}{2}\operatorname{tr}\left(\frac{\omega_2\mathbf{P}^2}{4\rho p\theta}\right) + \frac{1}{3MR}\left(\frac{2\theta_2|\mathbf{q}|^2}{3MR\rho^2\theta^3}\right)\right\}^{\cdot} + \operatorname{div}\left\{\frac{\mathbf{q}}{\theta} + \frac{\omega_3\mathbf{P}\mathbf{q}}{3MR\rho\theta^2}\right\} \\
&\quad - \hat{\omega}^4\frac{\partial}{\partial x_k}\left[\frac{\mu^3}{MR\rho^2}\frac{1}{2\mu\theta}P^{ij}\left(\frac{1}{2\mu\theta}P^{ij}\right)_{,k}\right] \\
&\quad - \hat{\theta}_4\frac{\partial}{\partial x_k}\left[\frac{\mu^3}{\rho^2}\left(\frac{2}{3MR\mu\theta}\right)q_i\left(\frac{2}{3MR\mu\theta}q_i\right)_{,k}\right] \\
&= -\frac{1}{2}\frac{\operatorname{tr}\mathbf{P}^2}{\mu\theta} - \frac{2}{3MR}\frac{|\mathbf{q}|^2}{\mu\theta^2} + \frac{2\omega_2 - \omega_6}{4p\theta}\operatorname{tr}(\mathbf{S}\mathbf{p}^2) \\
&\quad + \frac{2}{3MR}\left(\frac{2\theta_2}{3MR\rho\theta^3}\right)\operatorname{tr}(\mathbf{S}\mathbf{q}\otimes\mathbf{q})
\end{aligned}
$$

$$+\frac{1}{3MR}\left(-\overline{\theta_5}-\omega_3+\omega_3'(\theta)\,\theta\right)\operatorname{tr}(\mathbf{P}\operatorname{grad}\theta\otimes\mathbf{q})$$

$$+\left\{\hat{\omega}_2\,\frac{\operatorname{tr}\mathbf{S}^2}{p^2}+\hat{\omega}_3\,\frac{|\operatorname{grad}\theta|^2}{R\rho^2\theta^3}\right\}\frac{\mu}{R\theta}\operatorname{tr}\mathbf{P}^2$$

$$+\left\{\hat{\theta}_2\,\frac{\operatorname{tr}\mathbf{S}^2}{p^2}+\hat{\theta}_3\,\frac{|\operatorname{grad}\theta|^2}{R\rho^2\theta^3}\right\}\mu\,\frac{|\mathbf{q}|^2}{\theta^2}\left(\frac{2}{3MR}\right)$$

$$-\hat{\omega}_4\,\frac{\mu^3}{MR\rho^2}\left(\frac{1}{2\mu\theta}\,P^{ij}\right)_{,k}\left(\frac{1}{2\mu\theta}\,P^{ij}\right)_{,k}$$

$$-\hat{\theta}_4\,\frac{\mu^3\theta}{\rho^2}\left(\frac{2q_i}{3MR\mu\theta^2}\right)_{,k}\left(\frac{2q_i}{3MR\mu\theta^2}\right)_{,k} \tag{4.2}$$

Proof. From (1.7),

$$\rho\theta\dot{\eta}=-\operatorname{tr}(\mathbf{PS})-\operatorname{div}\mathbf{q}$$

This gives

$$-\rho\dot{\eta}=\operatorname{tr}\frac{\mathbf{PS}}{\theta}+\operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right)+\frac{\mathbf{q}\cdot\operatorname{grad}\theta}{\theta^2} \tag{4.3}$$

One must also compute $\operatorname{tr}\frac{\mathbf{PS}}{\theta}+\frac{\mathbf{q}\cdot\operatorname{grad}\theta}{\theta^2}$. To do this, note from (3.7), (3.10) that

$$-2\mu\mathbf{S}=\mathbf{P}^{eq}-\mathbf{P}_2-\mathbf{P}_3$$

$$-\tfrac{3}{2}\,\mu MR\operatorname{grad}\theta=\mathbf{q}^{eq}-\mathbf{q}_2-\mathbf{q}_3$$

Using (3.1), (3.2), one obtains

$$-2\mu\mathbf{S}=\frac{\omega_2\mu}{2p}\left(\dot{\mathbf{P}}-\mathbf{LP}-\mathbf{PL}^T+\frac{2}{3}\operatorname{tr}(\mathbf{PL}^T)\,\mathbf{I}\right)+\mathbf{P}-\mathbf{P}_2-\mathbf{P}_3 \tag{4.4}$$

$$-\frac{3}{2}\,\mu MR\operatorname{grad}\theta=\frac{2\theta_2\mu}{3MR\rho\theta}\left(\dot{\mathbf{q}}-\mathbf{L}^T\mathbf{q}\right)+\mathbf{q}-\mathbf{q}_2-\mathbf{q}_3 \tag{4.5}$$

Now substitute relations (4.4), (4.5) into (4.3) to obtain

$$
-\rho\dot\eta - \operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right)
$$

$$
= -\frac{1}{2\mu\theta}\operatorname{tr}\left\{\mathbf{P}\left[\frac{\omega_2\mu}{2p}\left(\dot{\mathbf{P}} - \mathbf{LP} - \mathbf{PL}^T + \frac{2}{3}\operatorname{tr}(\mathbf{PL}^T)\,\mathbf{I}\right) + \mathbf{P} - \mathbf{P}_2 - \mathbf{P}_3\right]\right\}
$$

$$
-\frac{\mathbf{q}}{\frac{3}{2}MR\mu\theta^2}\cdot\left[\frac{2\theta_2\mu}{3MR\rho\theta}(\dot{\mathbf{q}} - \mathbf{L}^T\mathbf{q}) + \mathbf{q} - \mathbf{q}_2 - \mathbf{q}_3\right] \tag{4.6}
$$

Next, note that

(i) $\operatorname{tr}\mathbf{P} = 0$;

(ii) $\operatorname{tr}(\mathbf{P}(\mathbf{LP} + \mathbf{PL}^T)) = 2\operatorname{tr}(\mathbf{SP}^2) + \frac{2}{3}(\operatorname{div}\mathbf{u})\operatorname{tr}\mathbf{P}^2$;

(iii) $\frac{1}{2}\rho\left(\frac{\operatorname{tr}\mathbf{P}^2\omega_2}{4\rho p\theta}\right)^{\!\boldsymbol{\cdot}} = \frac{\operatorname{tr}(\mathbf{PP}^{\boldsymbol{\cdot}})\,\omega_2}{4p\theta} + \frac{\operatorname{tr}\mathbf{P}^2}{4p\theta}\left[\dot\theta\left(\frac{\omega_2'(\theta)}{2} - \frac{\omega_2(\theta)}{\theta}\right) + \omega_2\operatorname{div}\mathbf{u}\right]$;

(iv) $\rho\left(\frac{|\mathbf{q}|^2\theta_2}{3MR\rho^2\theta^3}\right)^{\!\boldsymbol{\cdot}} = \frac{2\dot{\mathbf{q}}\cdot\mathbf{q}\theta_2}{3MR\rho\theta^3} + \left(\frac{2|\mathbf{q}|^2}{3MR\rho\theta^3}\right)\left[\dot\theta\left(\frac{\theta_2'(\theta)}{2} - \frac{3\theta_2}{2\theta}\right) + \theta_2\operatorname{div}\mathbf{u}\right]$;

(v) $\mathbf{q}\cdot\mathbf{L}^T\mathbf{q} = \operatorname{tr}(\mathbf{Sq}\otimes\mathbf{q}) + \frac{1}{3}(\operatorname{div}\mathbf{u})\,|\mathbf{q}|^2$.

In (iii) and (iv) we used $\dot\rho + \rho\operatorname{div}\mathbf{u} = 0$. Hence (4.6) may be rewritten as

$$
\rho\left\{-\eta + \frac{1}{2}\operatorname{tr}\left(\frac{\mathbf{P}^2\omega_2}{4\rho p\theta}\right) + \frac{1}{3MR}\left(\frac{2|\mathbf{q}|^2\theta_2}{3MR\rho^2\theta^3}\right)\right\}^{\!\boldsymbol{\cdot}}
$$

$$
= \operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right) + \frac{\operatorname{tr}\mathbf{P}^2}{4p\theta}\left[\dot\theta\left(\frac{\omega_2'(\theta)}{2} - \frac{\omega_2(\theta)}{\theta}\right) + \frac{5}{3}\omega_2\operatorname{div}\mathbf{u}\right]
$$

$$
+ \frac{2}{3MR}\left(\frac{2|\mathbf{q}|^2}{3MR\rho\theta^3}\right)\left[\dot\theta\left(\frac{\theta_2'(\theta)}{2} - \frac{3\theta_2}{2\theta}\right) + \frac{4}{3}\theta_2\operatorname{div}\mathbf{u}\right]
$$

$$
+ \frac{2\omega_2}{4p\theta}\operatorname{tr}(\mathbf{SP}^2) + \frac{2}{3MR}\left(\frac{2\theta_2}{3M\rho\theta^3}\right)\operatorname{tr}(\mathbf{Sq}\otimes\mathbf{q}) - \frac{\operatorname{tr}\mathbf{P}^2}{2\mu\theta} - \frac{2|\mathbf{q}|^2}{3MR\mu\theta^2}
$$

$$
+ \frac{1}{2\mu\theta}\operatorname{tr}[\mathbf{P}(\mathbf{P}_2 + \mathbf{P}_3)] + \frac{2}{3MR\mu\theta^2}\mathbf{q}(\mathbf{q}_2 + \mathbf{q}_3) \tag{4.7}
$$

Next we substitute relation (3.8), (3.11) for $\mathbf{P}_2$, $\mathbf{q}_2$ into (4.7). This yields

$$\rho\left\{-\eta+\frac{1}{2}\operatorname{tr}\left(\frac{\mathbf{P}^2\omega_2}{4\rho p\theta}\right)+\frac{1}{3MR}\left(\frac{2\,|\mathbf{q}|^2\,\theta_2}{3MR\rho^2\theta^2}\right)\right\}^{\cdot}$$
$$=\operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right)+\frac{\operatorname{tr}\mathbf{P}^2}{4p\theta}\left[\dot\theta\left(\frac{\omega_2'(\theta)}{2}-\frac{\omega_2(\theta)}{\theta}+\frac{\mu'(\theta)}{\mu(\theta)}\,\omega_2\right)+\operatorname{div}\mathbf{u}\left(\frac{5}{3}\,\omega_2-\omega_1\right)\right]$$
$$+\frac{2}{3MR}\left(\frac{2\,|\mathbf{q}|^2}{3MR\rho\theta^3}\right)\left[\dot\theta\left(\frac{\theta_2'(\theta)}{2}-\frac{3\theta_2}{2\theta}+\frac{\mu'(\theta)}{\mu(\theta)}\,\theta_2\right)+\operatorname{div}\mathbf{u}\left(\frac{4}{3}\,\theta_2-\theta_1\right)\right]$$
$$+\frac{2\omega_2}{4p\theta}\operatorname{tr}(\mathbf{S}\mathbf{P}^2)+\frac{2}{3MR}\,\frac{2\theta_2}{3M\rho\theta^3}\operatorname{tr}(\mathbf{S}\mathbf{q}\otimes\mathbf{q})-\frac{\operatorname{tr}\mathbf{P}^2}{2\mu\theta}-\frac{2\,|\mathbf{q}|^2}{3MR\mu\theta^2}$$
$$+\frac{1}{2\mu\theta}\operatorname{tr}[\mathbf{P}(\tilde{\mathbf{P}}_2+\mathbf{P}_3)]+\frac{2}{3MR\mu\theta^2}\,\mathbf{q}\cdot(\tilde{\mathbf{q}}_2+\mathbf{q}_3) \tag{4.8}$$

where

$$\tilde{\mathbf{P}}_2=\mathbf{P}_2+\mu\,\frac{\omega_1\operatorname{div}\mathbf{u}}{2p}\,\mathbf{P}-\omega_2\,\frac{\mu'(\theta)\,\dot\theta}{2p}\,\mathbf{P}$$
$$\tilde{\mathbf{q}}_2=\mathbf{q}_2+\mu\,\frac{2\theta_1\operatorname{div}\mathbf{u}}{3MR\rho\theta}\,\mathbf{q}-\frac{2\theta_2\dot\theta\mu'(\theta)}{3MR\rho\theta}\,\mathbf{q}$$

Notice however that

$$\dot\theta\left(\frac{\omega_2'(\theta)}{2}-\frac{\omega_2(\theta)}{\theta}+\frac{\mu'(\theta)}{\mu(\theta)}\,\omega_2\right)+\operatorname{div}\mathbf{u}\left(\frac{5}{3}\,\omega_2-\omega_1\right)$$
$$=\left(\dot\theta+\frac{2}{3}\,\theta\operatorname{div}\mathbf{u}\right)\left(\frac{\omega_2'(\theta)}{2}-\frac{\omega_2(\theta)}{\theta}+\frac{\mu'(\theta)}{\mu(\theta)}\,\omega_2\right)$$
$$+\theta\operatorname{div}\mathbf{u}\left[-\frac{2}{3}\left(\frac{\omega_2'(\theta)}{2}-\frac{\omega_2(\theta)}{\theta}+\frac{\mu'(\theta)}{\mu(\theta)}\,\omega_2\right)+\frac{5}{3}\frac{\omega_2}{\theta}-\frac{\omega_1}{\theta}\right]$$
$$=\left(\dot\theta+\frac{2}{3}\,\theta\operatorname{div}\mathbf{u}\right)\left(\frac{\omega_2'(\theta)}{2}-\frac{\omega_2(\theta)}{\theta}+\frac{\mu'(\theta)}{\mu(\theta)}\,\omega_2\right)$$
$$+\theta\operatorname{div}\mathbf{u}\left[\frac{7}{3}\frac{\omega_2(\theta)}{\theta}-\frac{\omega_2'(\theta)}{3}-\frac{2}{3}\frac{\mu'(\theta)}{\mu(\theta)}\,\omega_2-\frac{\omega_1}{\theta}\right]$$
$$=\left(\dot\theta+\frac{2}{3}\,\theta\operatorname{div}\mathbf{u}\right)\left(\frac{\omega_2'(\theta)}{2}-\frac{\omega_2(\theta)}{\theta}+\frac{\mu'(\theta)}{\mu(\theta)}\,\omega_2\right)$$

where (2.8) has been used. Similarly,

$$
\begin{aligned}
&\dot{\theta}\left(\frac{\theta_2'(\theta)}{2}-\frac{3\theta_2}{2}+\frac{\mu'(\theta)}{\mu(\theta)}\,\theta_2\right)+\operatorname{div}\mathbf{u}\left(\frac{4}{3}\,\theta_2-\theta_1\right)\\
&\quad=\left(\dot{\theta}+\frac{2}{3}\,\theta\operatorname{div}\mathbf{u}\right)\left(\frac{\theta_2'(\theta)}{2}-\frac{3\theta_2}{2\theta}+\frac{\mu'(\theta)}{\mu(\theta)}\,\theta_2\right)\\
&\qquad+\theta\operatorname{div}\mathbf{u}\left[-\frac{2}{3}\left(\frac{\theta_2'(\theta)}{2}-\frac{3\theta_2}{2\theta}+\frac{\mu'(\theta)}{\mu(\theta)}\,\theta_2\right)+\frac{4}{3}\frac{\theta_2}{\theta}-\frac{\theta_1}{\theta}\right]\\
&\quad=\left(\dot{\theta}+\frac{2}{3}\,\theta\operatorname{div}\mathbf{u}\right)\left(\frac{\theta_2'(\theta)}{2}-\frac{3\theta_2}{2\theta}+\frac{\mu'(\theta)}{\mu(\theta)}\,\theta_2\right)\\
&\qquad+\theta\operatorname{div}\mathbf{u}\left[\frac{7\theta_2}{3\theta}-\frac{1}{3}\,\theta_2'(\theta)-\frac{2}{3}\frac{\mu'(\theta)}{\mu(\theta)}\,\theta_2-\frac{\theta_1}{\theta}\right]\\
&\quad=\left(\dot{\theta}+\frac{2}{3}\,\theta\operatorname{div}\mathbf{u}\right)\left(\frac{\theta_2'(\theta)}{2}-\frac{3\theta_2}{2\theta}+\frac{\mu'(\theta)}{\mu(\theta)}\,\theta_2\right)
\end{aligned}
$$

where again (2.8) was used. Hence, (4.8) simplifies to

$$
\begin{aligned}
&\rho\left\{-\eta+\frac{1}{2}\operatorname{tr}\left(\frac{\mathbf{P}^2\omega_2}{4\rho p\theta}\right)+\frac{1}{3MR}\left(\frac{2\,|\mathbf{q}|^2\,\theta_2}{3MR\rho^2\theta^3}\right)\right\}^{\cdot}\\
&\quad=\operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right)+\frac{\operatorname{tr}\mathbf{P}^2}{4p\theta}\left(\dot{\theta}+\frac{2}{3}\,\theta\operatorname{div}\mathbf{u}\right)\left(\frac{\omega_2'(\theta)}{2}-\frac{\omega_2(\theta)}{\theta}+\frac{\mu'(\theta)}{\mu(\theta)}\,\omega_2\right)\\
&\qquad+\frac{2}{3MR}\left(\frac{2\,|\mathbf{q}|^2}{3MR\rho\theta^3}\right)\left(\dot{\theta}+\frac{2}{3}\,\theta\operatorname{div}\mathbf{u}\right)\left(\frac{\theta_2'(\theta)}{2}-\frac{3\theta_2}{2\theta}+\frac{\mu'(\theta)}{\mu(\theta)}\,\theta_2\right)\\
&\qquad+\frac{2\omega_2}{4p\theta}\operatorname{tr}(\mathbf{S}\mathbf{P}^2)+\frac{2}{3MR}\left(\frac{2\theta_2}{3MR\rho\theta^3}\right)\operatorname{tr}(\mathbf{S}\mathbf{q}\otimes\mathbf{q})\\
&\qquad-\frac{\operatorname{tr}\mathbf{P}^2}{2\mu\theta}-\frac{2\,|\mathbf{q}|^2}{3MR\mu\theta^2}+\frac{1}{2\mu\theta}\operatorname{tr}[\mathbf{P}(\tilde{\mathbf{P}}_2+\mathbf{P}_3)]+\frac{2}{3MR\mu\theta^2}\,\mathbf{q}\cdot(\tilde{\mathbf{q}}_2+\mathbf{q}_3)
\end{aligned}
\tag{4.9}
$$

Next note the following identity:

$$\frac{1}{3MR}\operatorname{div}\left(\frac{\omega_3 \mathbf{Pq}}{\rho\theta^2}\right) = -\frac{\omega_3}{\rho^2\theta^3}\operatorname{tr}(\mathbf{P}\operatorname{grad}(\rho\theta)\otimes\mathbf{q})$$

$$+\frac{\omega_3\operatorname{div}(\mathbf{Pq})}{3MR\rho\theta^2}+\frac{\omega_3'(\theta)-\dfrac{\omega_3(\theta)}{\theta}}{3MR\rho\theta^2}\operatorname{tr}(\mathbf{P}\operatorname{grad}\theta\otimes\mathbf{q}) \qquad (4.10)$$

Substitute the definitions of $\tilde{\mathbf{q}}_2$, $\tilde{\mathbf{P}}_2$ and the relation (4.10) into the expression on the left hand side of the following equations, one gets

$$\frac{\operatorname{tr}(\mathbf{P}\tilde{\mathbf{P}}_2)}{2\mu\theta}+\frac{2}{3MR\mu\theta^2}\mathbf{q}\cdot\tilde{\mathbf{q}}_2+\frac{1}{3MR}\operatorname{div}\frac{\omega_3\mathbf{Pq}}{\rho\theta^2}$$

$$=\frac{-1}{3MR\rho^2\theta^3}\operatorname{tr}(\mathbf{P}\operatorname{grad}(\rho\theta)\otimes\mathbf{q})(\omega_3+\omega_4+\theta_3)-\frac{\operatorname{div}(\mathbf{Pq})}{3MR\rho\theta^2}(-\omega_3+\theta_4)$$

$$+\frac{1}{3MR\rho\theta^3}\operatorname{tr}(\mathbf{P}\operatorname{grad}\theta\otimes\mathbf{q})\left(\omega_3'(\theta)\,\theta-\omega_5-\theta_5+\frac{\mu'(\theta)\,\theta}{\mu(\theta)}(\omega_3+\omega_4)-\omega_3\right)$$

$$-\frac{\omega_6}{4\theta p}\operatorname{tr}(\mathbf{P}^2\mathbf{S}) \qquad (4.11)$$

But if we use identities (2.8), (2.11) to see that

$$\omega_3+\omega_4+\theta_3=0, \qquad \omega_3=\theta_4$$

$$\omega_3'(\theta)\,\theta-\omega_3-\omega_5-\theta_5+\frac{\mu'(\theta)\,\theta}{\mu(\theta)}(\omega_3+\omega_4)=-\overline{\theta_5}-\omega_3+\omega_3'(\theta)\,\theta$$

then (4.11) simplifies to

$$\frac{\operatorname{tr}(\mathbf{P}\tilde{\mathbf{P}}_2)}{2\mu\theta}+\frac{2}{3MR\mu\theta^2}\mathbf{q}\cdot\tilde{\mathbf{q}}_2+\frac{1}{3MR}\operatorname{div}\left(\frac{\omega_3\mathbf{Pq}}{\rho\theta^2}\right)$$

$$=\frac{1}{3MR\rho\theta^3}\operatorname{tr}(\mathbf{P}\operatorname{grad}\theta\otimes\mathbf{q})(\omega_3'(\theta)\,\theta-\omega_3-\overline{\theta_5})-\frac{\omega_6}{4\theta p}\operatorname{tr}(\mathbf{P}^2\mathbf{S}) \qquad (4.12)$$

Finally, substitute the relations for

$$\frac{1}{2\mu\theta}\operatorname{tr}[\mathbf{P}(\tilde{\mathbf{P}}_2+\mathbf{P}_3)]+\frac{2}{3MR\mu\theta^2}\mathbf{q}\cdot(\tilde{\mathbf{q}}_2+\mathbf{q}_3)$$

obtained from (4.12) and the definitions of $\mathbf{P}_3$, $\mathbf{q}_3$ given by (3.9), (3.12), (4.1) into (4.9) to obtain the desired identity (4.2). This completes the proof

of the Lemma.

We now majorize the indeterminate terms on the right side of (4.2). First recall that for either $\mathbf{S}$ or $\mathbf{P}$ the Cayley–Hamilton Theorem implies:

$$-\mathbf{S}^3 + I_1\mathbf{S}^2 - I_2\mathbf{S} + I_3\mathbf{I} = 0$$

where

$$I_1 = -\operatorname{tr}\mathbf{S}, \qquad I_2 = \tfrac{1}{2}\left[(tr\,\mathbf{S})^2 - tr\,\mathbf{S}^2\right], \qquad I_3 = \det\mathbf{S}$$

Since $tr\,\mathbf{S} = 0$ we have

$$-\mathbf{S}^3 + (\tfrac{1}{2}tr\,\mathbf{S}^2)\,\mathbf{S} + (\det\mathbf{S})\,\mathbf{I} = 0$$

Hence $-tr\,\mathbf{S}^3 + 3\det\mathbf{S} = 0$, which implies

$$\mathbf{S}^3 = \tfrac{1}{2}(tr\,\mathbf{S}^2)\,\mathbf{S} + \tfrac{1}{3}(tr\,\mathbf{S}^3)\,\mathbf{I}$$

and

$$\mathbf{S}^4 = \tfrac{1}{2}(tr\,\mathbf{S}^2)\,\mathbf{S}^2 + \tfrac{1}{3}(tr\,\mathbf{S}^3)\,\mathbf{S}$$

This in turn gives

$$tr\,\mathbf{S}^4 = \tfrac{1}{2}(tr\,\mathbf{S}^2)^2 \tag{4.13}$$

and similarly

$$tr\,\mathbf{P}^4 = \tfrac{1}{2}(tr\,\mathbf{P}^2)^2 \tag{4.14}$$

By the Cauchy–Schwartz inequality

$$|\operatorname{tr}(\mathbf{S}\mathbf{P}^2)| \leqslant (\operatorname{tr}\mathbf{S}^2)^{1/2}(\operatorname{tr}\mathbf{P}^4)^{1/2}$$

and from (4.14),

$$|\operatorname{tr}(\mathbf{S}\mathbf{P}^2)| \leqslant \frac{1}{\sqrt{2}}((\operatorname{tr}\mathbf{S}^2)(\operatorname{tr}\mathbf{P}^2))^{1/2}(\operatorname{tr}\mathbf{P}^2)^{1/2} \tag{4.15}$$

Next, note

$$|\mathrm{tr}(\mathbf{S}\mathbf{q}\otimes\mathbf{q})| = |\mathbf{S}\mathbf{q}\cdot\mathbf{q}| \leqslant |\mathbf{S}\mathbf{q}|\,|\mathbf{q}| \leqslant ((\mathrm{tr}\,\mathbf{S}^2)^{1/2}\,|\mathbf{q}|)\,|\mathbf{q}| \tag{4.16}$$

$$|\mathrm{tr}(\mathbf{P}\,\mathrm{grad}\,\theta\otimes\mathbf{q})| = |\mathbf{P}\,\mathrm{grad}\,\theta\cdot\mathbf{q}| \leqslant |\mathbf{P}\,\mathrm{grad}\,\theta|\,|\mathbf{q}| \leqslant ((\mathrm{tr}\,\mathbf{P}^2)^{1/2}\,|\mathrm{grad}\,\theta|)\,|\mathbf{q}| \tag{4.17}$$

Now use (4.15), (4.16), (4.17) to majorize the right hand side of (4.2) to obtain the following entropy inequality:

$$
\begin{aligned}
&\rho\left\{-\eta + \frac{1}{2}\mathrm{tr}\left(\frac{\omega_2\mathbf{P}^2}{4\rho p\theta}\right) + \frac{1}{3MR}\left(\frac{2\theta_2\,|\mathbf{q}|^2}{3MR\rho^2\theta^3}\right)\right\}^{\cdot} + \mathrm{div}\left\{\frac{\mathbf{q}}{\theta} + \frac{\omega_3\mathbf{P}\mathbf{q}}{3MR\rho\theta^2}\right\}\\
&\quad -\hat{\omega}_4\frac{\partial}{\partial x_k}\left[\frac{\mu^3}{MR\rho^2}\frac{1}{2\mu\theta}P^{ij}\left(\frac{1}{2\mu\theta}P^{ij}\right)_{,k}\right]\\
&\quad -\hat{\theta}_4\frac{\partial}{\partial x_k}\left[\frac{\mu^3\theta}{\rho^2}\left(\frac{2}{3MR\mu\theta^2}q_i\right)\left(\frac{2}{3MR\mu\theta^2}q_i\right)_{,k}\right]\\
&\leqslant -\frac{1}{2}\frac{\mathrm{tr}\,\mathbf{P}^2}{\mu\theta} - \frac{2}{3MR}\frac{|\mathbf{q}|^2}{\mu\theta^2} + \frac{|2\omega_2-\omega_6|}{4\sqrt{2}\,p\theta}((\mathrm{tr}\,\mathbf{S}^2)(\mathrm{tr}\,\mathbf{P}^2))^{1/2}(\mathrm{tr}\,\mathbf{P}^2)^{1/2}\\
&\quad + \frac{2}{3MR}\left(\frac{2\theta_2}{3MR\rho\theta^3}\right)((\mathrm{tr}\,\mathbf{S}^2)^{1/2}\,|\mathbf{q}|)\,|\mathbf{q}|\\
&\quad + \frac{1}{3MR\rho\theta^3}|-\overline{\theta_5}-\omega_3+\omega_3'(\theta)\,\theta|\,(\mathrm{tr}\,\mathbf{P}^2)^{1/2}\,|\mathrm{grad}\,\theta|\,|\mathbf{q}|\\
&\quad + \left\{\hat{\omega}_2\frac{\mathrm{tr}\,\mathbf{S}^2}{p^2} + \hat{\omega}_3\frac{|\mathrm{grad}\,\theta|^2}{R\rho^2\theta^3}\right\}\frac{\mu}{R\theta}\mathrm{tr}\,\mathbf{P}^2\\
&\quad + \left\{\hat{\theta}_2\frac{\mathrm{tr}\,\mathbf{S}^2}{p^2} + \hat{\theta}_3\frac{|\mathrm{grad}\,\theta|^2}{R\rho^2\theta^3}\right\}\mu\frac{|\mathbf{q}|^2}{\theta^2}\left(\frac{2}{3MR}\right) \tag{4.18}
\end{aligned}
$$

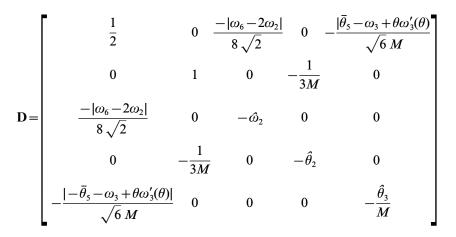This proves the following theorem on the entropy inequality.

Theorem 4.2. In addition to the hypothesis of Lemma 4.1, we assume that $\hat{\omega}_4\geqslant 0$, $\hat{\theta}_4\geqslant 0$. Furthermore, if we define $\mathbf{z}\in\mathbb{R}^5$ by

$$
\begin{aligned}
\mathbf{z} = \Bigg[&\left(\frac{tr\,\mathbf{P}^2}{\theta}\right)^{1/2}, \sqrt{\frac{2}{3}}\frac{|\mathbf{q}|}{\sqrt{MR}\,\theta}, \mu\frac{(tr\,\mathbf{S}^2 tr\,\mathbf{P}^2)^{1/2}}{p(R\theta)^{1/2}}, \mu\sqrt{\frac{2}{3}}\frac{(tr\,\mathbf{S}^2)^{1/2}\,|\mathbf{q}|}{p\theta(MR)^{1/2}},\\
&\sqrt{\frac{2}{3}}\frac{|\mathrm{grad}|\,\theta|\,\mathbf{q}|}{p\theta^{3/2}}\Bigg]
\end{aligned}
$$

then the following entropy inequality holds:

$$
\rho \left\{ -\eta + \frac{1}{2}\operatorname{tr}\left(\frac{\omega_2 \mathbf{P}^2}{4\rho p\theta}\right) + \frac{1}{3MR}\left(\frac{2\theta_2 |\mathbf{q}|^2}{3MR\rho^2\theta^3}\right) \right\}^{\cdot} + \operatorname{div}\left\{\frac{\mathbf{q}}{\theta} + \frac{\omega_3 \mathbf{Pq}}{3MR\rho\theta^2}\right\}
$$

$$
-\hat{\omega}_4 \frac{\partial}{\partial x_k}\left[\frac{\mu^3}{MR\rho^2}\frac{1}{2\mu\theta}P^{ij}\left(\frac{1}{2\mu\theta}P^{ij}\right)_{,k}\right]
$$

$$
-\hat{\theta}_4 \frac{\partial}{\partial x_k}\left[\frac{\mu^3}{\rho^2}\left(\frac{2}{3MR\mu\theta}\right)q_i\left(\frac{2}{3MR\mu\theta}q_i\right)_{,k}\right]
$$

$$
\leqslant -\frac{1}{\mu}\mathbf{z}\cdot\mathbf{Dz} \tag{4.19}
$$

where

$$
\mathbf{D} = \begin{bmatrix}
\dfrac{1}{2} & 0 & \dfrac{-|\omega_6 - 2\omega_2|}{8\sqrt{2}} & 0 & -\dfrac{|\bar{\theta}_5 - \omega_3 + \theta\omega_3'(\theta)|}{\sqrt{6}\,M} \\[2ex]
0 & 1 & 0 & -\dfrac{1}{3M} & 0 \\[2ex]
\dfrac{-|\omega_6 - 2\omega_2|}{8\sqrt{2}} & 0 & -\hat{\omega}_2 & 0 & 0 \\[2ex]
0 & -\dfrac{1}{3M} & 0 & -\hat{\theta}_2 & 0 \\[2ex]
-\dfrac{|-\bar{\theta}_5 - \omega_3 + \theta\omega_3'(\theta)|}{\sqrt{6}\,M} & 0 & 0 & 0 & -\dfrac{\hat{\theta}_3}{M}
\end{bmatrix}
$$

D is positive definite if $\hat{\omega}_2 < 0$, $\hat{\theta}_2 < 0$, $\hat{\theta}_3 < 0$ are sufficiently large in absolute value, $\hat{\omega}_3 \leqslant 0$, and $|-\bar{\theta}_5 - \omega_3 + \theta\omega_3'(\theta)|$, $|\omega_6 - 2\omega_3|$ are bounded.

Remark 1. In Theorem 4.2 the positive definiteness of *D* is a sufficient condition but may not be necessary. Currently we are not able to prove the necessary condition.

Remark 2. If $\hat{\omega}_4 = \hat{\theta}_4 = 0$, namely, the dissipative terms in $\mathbf{P}_3$ and $\mathbf{q}_3$ are not present, the entropy condition still holds and the entropy and the entropy flux in (4.19) agree with those of Grad's thirteen moment theory.[19] The generalized entropy, as in Grad's theory, is not globally

convex. However, it is locally convex around the equilibrium solution (ρ and θ are constants), thus the rest state ($u=0$) is stable, in contrast to the Burnett equations where the rest state is unstable.

5. HYPERBOLICITY

In this section we study the hyperbolicity of the relaxation approximation (3.1)–(3.6). Our computation will show that locally, for a one dimensional motion, the relaxation system (3.1)–(3.6) given below is hyperbolic when the parabolic like contributions given via $\hat{\omega}_4$ and $\hat{\theta}_4$ are omitted (i.e. $\hat{\omega}_4 = \hat{\theta}_4 = 0$). Inclusion of these terms makes our system weakly parabolic. In order to reduce the system to the one dimensional case, we assume that all quantities depend on x only,

$$\mathbf{u} = (u(x, t), 0, 0)$$

and look for special solution

$$P^{23} = P^{13} = P^{12} = q_2 = q_3 = 0$$

it is easy to show that these are exact solutions to (3.4) and (3.6). Furthermore, one can show that

$$P^{22} = P^{33}$$

is also consistent with (3.4) and (3.6). Since **P** has zero trace, this implies that

$$P^{22} = -\tfrac{1}{2} P^{11}$$

Thus we are left with five independent variables ρ, u, θ, $P^{11} = \sigma$ and $q_1 = q$, satisfying the system

$$\rho_t + u\rho_x + \rho u_x = 0 \tag{5.1}$$

$$u_t + uu_x + \frac{1}{\rho} p_x + \frac{1}{\rho} \sigma_x = b \tag{5.2}$$

$$\theta_t + u\theta_x + \frac{2p}{3\rho R} u_x + \frac{2}{3\rho R} \sigma u_x + \frac{2}{3\rho R} q_x = 0 \tag{5.3}$$

$$\sigma_t + u\sigma_x - \frac{4}{3} \sigma u_x = -\frac{2p}{\omega_2 \mu} (\sigma - \sigma_{eq}) \tag{5.4}$$

$$q_t + uq_x - qu_x = -\frac{3Mp}{2\theta_2 \mu} (q - q_{eq}) \tag{5.5}$$

where

$$\sigma_{eq} = -\frac{4}{3}\mu u_x + \sigma_2 + \sigma_3 \tag{5.6}$$

$$\sigma_2 = -\mu \frac{\omega_1}{2p}\sigma u_x + \frac{\omega_2 \mu'(\theta)\dot{\theta}}{2p}\sigma - \mu^2 \frac{4\omega_3}{9\rho\theta}\left(\frac{q}{\mu MR}\right)_x$$
$$-\mu \frac{4\omega_4}{9\rho p\theta}\frac{q}{MR}p_x - \mu \frac{4\omega_5}{9\rho\theta^2}\frac{q}{MR}\theta_x - \mu\frac{\omega_6}{6p}\sigma u_x \tag{5.7}$$

$$\sigma_3 = \mu^2\left[\frac{2\hat{\omega}_2}{3p^2}u_x{}^2 + \hat{\omega}_3 \frac{\theta_x{}^2}{R\rho^2\theta^3}\right]\sigma - \mu\frac{\hat{\gamma}_1}{p\theta}\frac{2}{3\rho R}(\sigma^2 u_x + \sigma q_x)$$
$$-\hat{\omega}_4\left[\frac{\mu^3}{MR\rho^2}\left(\frac{\sigma}{2\mu\theta}\right)_x\right]_x \tag{5.8}$$

$$q_{eq} = -\frac{3}{2}\mu MR\theta_x + q_2 + q_3 \tag{5.9}$$

$$q_2 = -2\mu\frac{\theta_1}{3MR\rho\theta}qu_x + \frac{2\theta_2\dot{\theta}\mu'(\theta)}{3MR\rho\theta}q$$
$$-\mu\frac{\theta_3}{2p\rho}\sigma p_x - \mu^2\frac{\theta_4}{2\rho}\left(\frac{\sigma}{\mu}\right)_x - \mu\frac{\theta_5}{2\rho\theta}\sigma\theta_x \tag{5.10}$$

$$q_3 = \mu^2\left[\frac{2\hat{\theta}_2}{3p^2}u_x{}^2 + \hat{\theta}_3 \frac{\theta_x{}^2}{R\rho^2\theta^3}\right]q - \mu\frac{\hat{\lambda}_1}{\rho\theta^2}\frac{2}{3\rho R}(\sigma u_x + q_x)\left(\frac{q}{\frac{3}{2}MR}\right)$$
$$-\hat{\theta}_4\left[\frac{\mu^3\theta}{\rho^2}\left(\frac{2}{3MR\mu\theta^2}q\right)_x\right]_x \tag{5.11}$$

Set $\hat{\omega}_4 = \hat{\theta}_4 = 0$. Upon using $p = R\rho\theta$, and (5.3) to replace $\dot{\theta}$, one obtains the Jacobi matrix for the relaxation system (5.1)–(5.5):

$$\mathbf{J} = \begin{bmatrix} u & \rho & 0 & 0 & 0 \\ \dfrac{R\theta}{\rho} & u & R & \dfrac{1}{\rho} & o \\ 0 & J_{32} & u & 0 & \dfrac{2}{3R\rho} \\ J_{41} & J_{42} & J_{43} & u & J_{46} \\ J_{51} & J_{52} & J_{53} & J_{54} & J_{56} \end{bmatrix} \tag{5.12}$$

where

$$J_{32} = \frac{2}{3}\theta + \frac{2}{2R\rho}\sigma$$

$$J_{41} = \frac{8\omega_4 q}{9\omega_2 \rho M}$$

$$J_{42} = -\frac{4}{3}\sigma + \frac{8p}{3\omega_2} + \frac{\omega_1}{\omega_2}\sigma - 2\frac{\mu'}{\mu}\sigma\left(-\frac{1}{3}\theta - \frac{1}{3\rho R}\sigma\right)$$
$$+ \frac{\omega_6}{3\omega_2}\sigma + \frac{4\hat{\gamma}_1}{3\omega_2 p}\sigma^2$$

$$J_{43} = -\frac{8\omega_3 q\mu'}{9\omega_2 \mu M} + \frac{8(\omega_4 + \omega_5)q}{9\omega_2 \theta M}$$

$$J_{46} = \frac{2\mu'\sigma}{3\mu\rho R} + \frac{8\omega_3}{9\omega_2 M} + \frac{4\hat{\gamma}_1\sigma}{3\omega_2 p} \qquad (5.13)$$

$$J_{51} = \frac{3\theta_3 MR\theta}{4\theta_2 \rho}\sigma$$

$$J_{52} = -q + \frac{\theta_1}{\theta_2}q - \frac{\mu'}{\mu}q\left(-\frac{2}{3}\theta - \frac{2}{3\rho R}\sigma\right) + \frac{2\hat{\lambda}_1 q}{3\theta_2 p}\sigma$$

$$J_{53} = \frac{9M^2 pR}{4\theta_2} - \frac{3Mp\theta_4\mu'}{4\theta_2 \rho\mu}\sigma + \frac{3MR(\theta_3 + \theta_5)}{4\theta_2}\sigma$$

$$J_{54} = \frac{3Mp\theta_4}{4\theta_2 \rho}$$

$$J_{56} = u + \frac{2\mu' q}{3\mu\rho R} + \frac{2\hat{\lambda}_1 q}{3\theta_2 p}$$

The characteristic polynomial of the Jacobian matrix (5.12) is

$$(u - \lambda)^4 (j_{56} - \lambda) + (u - \lambda)^4 \left[-\frac{2}{3R\rho}J_{53} - J_{46}J_{54} \right]$$
$$+ (u - \lambda)^2 (j_{56} - \lambda) \left[-RJ_{32} - \frac{1}{\rho}J_{42} - R\theta \right]$$
$$+ (u - \lambda)^2 \left[\frac{2}{3R\rho}J_{43}J_{54} + \frac{2}{3\rho}J_{52} + \frac{1}{\rho}J_{46}J_{52} \right]$$

$$+(u-\lambda)^2(j_{56}-\lambda)\left[\frac{1}{\rho}J_{32}J_{43}+J_{41}\right]$$

$$+(u-\lambda)\left[-\frac{2}{3\rho}J_{42}J_{54}+RJ_{32}J_{54}+\frac{2}{3R\rho^2}J_{42}J_{53}-\frac{2}{3R\rho^2}J_{43}J_{52}\right.$$

$$\left.-\frac{1}{\rho}J_{32}J_{46}J_{53}-J_{46}J_{51}+R\theta J_{46}J_{54}-\frac{2}{3}J_{51}+\frac{2\theta}{3\rho}J_{53}\right]$$

$$+\left[-\frac{2\theta}{3\rho}J_{43}J_{54}-\frac{2}{3\rho R}J_{41}J_{53}+\frac{2}{3\rho R}J_{43}J_{51}+\frac{2}{3}J_{41}J_{54}\right] \tag{5.14}$$

For Maxwellian molecules $\mu/\mu'=\theta$, those quantities in (5.13) become

$$J_{32}=\frac{2}{3}\theta+\frac{2}{3R\rho}\sigma$$

$$J_{41}=0, \qquad J_{42}=\frac{7}{3}\sigma+\frac{4}{3}p+\frac{2}{3}\frac{\sigma^2}{p}, \qquad J_{43}=0, \qquad J_{46}=\frac{2}{3}\frac{\sigma}{p}+\frac{8}{15}$$

$$J_{51}=-\frac{p}{\rho^2}\sigma, \qquad J_{52}=\frac{q}{p}\sigma+\frac{4}{3}q, \qquad J_{53}=\frac{5}{2}pR+\frac{31}{4}R\sigma, \tag{5.15}$$

$$J_{54}=\frac{p}{\rho}, \qquad J_{56}=u+\frac{q}{p}$$

while the characteristic polynomial becomes (upon changing $u-\lambda\to-\lambda$):

$$\lambda[\lambda^4+a_3\lambda^3+a_2\lambda^2+a_1\lambda+a_0] \tag{5.16}$$

with the coefficients

$$a_0=\frac{123}{10}\frac{P\sigma}{\rho^2}+3\frac{p^2}{\rho^2}+\frac{587}{90}\frac{\sigma^2}{\rho^2}, \qquad a_1=\frac{7}{5}\frac{q}{\rho}+\frac{41}{45}\frac{q\sigma}{p\rho}$$

$$a_2=-\frac{26}{5}\frac{p}{\rho}-\frac{53}{6}\frac{\sigma}{\rho}-\frac{2}{3}\frac{\sigma^2}{p\rho}, \qquad a_3=-\frac{q}{p} \tag{5.17}$$

While it is very difficult to see whether the characteristic polynomial (5.16) has five real roots, we instead investigate the linearized system (5.1)–(5.5) around

$$(\bar{\rho}, 0, \bar{\theta}, 0, 0)$$

where $\bar{\rho}$, $\bar{\theta}$ are constants. In this case, the characteristic polynomial (5.16) reduces to

$$\lambda \left[\lambda^4 - \frac{26}{5} \frac{\bar{p}}{\bar{\rho}} \lambda^2 + 3 \left(\frac{\bar{p}}{\bar{\rho}} \right)^2 \right] \tag{5.18}$$

where $\bar{p} = R\bar{\theta}\bar{\rho}$. This polynomial has five distinct roots

$$0, \qquad \pm \sqrt{\frac{13}{5} \pm \sqrt{\frac{94}{25}}} \sqrt{\frac{\bar{p}}{\bar{\rho}}}$$

Thus the linearized relaxation system, when the parabolic terms are omitted, is hyperbolic. This shows that the relaxation system is at least locally well-posed for initial value problems. By incorporating the weakly parabolic terms we expect to get the global well-posedness but will leave this for future research.

NOMENCLATURE

b   body force
B   subset of Euclidean space
div   divergence
e   internal energy density
grad   gradient
I   unit tensor
L   velocity gradient ($\mathbf{L} = \text{grad } \mathbf{u}$)
M   Maxwell number
n   unit exterior normal
P   pressure deviator ($\mathbf{P} = [P^{ij}]_{3 \times 3}$)p mean normal pressure
q   energy flux vector ($\mathbf{q} = [q_1, q_2, q_3]^T$)
R   gas constant
S   distortion tensor ($\mathbf{S} = \frac{1}{2}(\text{grad } \mathbf{u} + (\text{grad } \mathbf{u})^T - \frac{2}{3} \text{div } \mathbf{uI}))$

T	stress tensor ($\mathbf{T} = -p\mathbf{I} - \mathbf{P}$)
t	time
tr	trace
u	macroscopic velocity
x	cartesian coordinate ($\mathbf{x} = (x, y, z)$)
μ	viscosity
ρ	mass density
η	specific entropy
ψ	Helmholtz free energy, $\psi = \varepsilon - \theta\eta$
θ	temperature
$\theta_i, \hat{\theta}_i$	coefficients of the Chapman–Enskog expansion for $\mathbf{q}$
$\omega_i, \hat{\omega}_i$	coefficients of the Chapman–Enskog expansion for $\mathbf{P}$
$\hat{\gamma}_1, \hat{\lambda}_1$	coefficients of the super Burnett terms
(˙)	material derivative of ( ), i.e., $(\dot{\ }) = \frac{\partial}{\partial t}(\ ) + \mathbf{u} \cdot \mathrm{grad}(\ )$
$\otimes$	dyadic product, i.e. $(\mathbf{u} \otimes \mathbf{v})_{ij} = u_i v_j$
·	inner product, i.e. $\mathbf{u} \cdot \mathbf{v} = u_i v_i$ for vectors $\mathbf{u}, \mathbf{v}$;
	$\mathbf{A} \cdot \mathbf{B} = tr(\mathbf{AB})$ for tensors $\mathbf{A}, \mathbf{B}$.

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